Imp

* Lagrange's Equation from D'Alembert's principle :-

Suppose the System in which position Coordinate are may be defined as

$$\vec{r_i} = \vec{r_i}(q_1, q_2, q_3 \ldots q_n \ldots t) \quad \text{--- (1)}$$

diff. w. r. to t

$$\frac{d\vec{r_i}}{dt} = \frac{\partial r_i}{\partial q_1}\frac{\delta q_1}{dt} + \frac{\delta r_i}{\partial q_2}\frac{\delta q_2}{\delta t} \ldots \frac{dr_i}{\partial q_n}\frac{\delta q_n}{dt} \ldots \frac{\delta r_i}{dt}$$

$$\Rightarrow \vec{v_i} = \sum_j \frac{\partial \vec{r_i}}{\partial q_j}\dot{q}_j + \frac{\partial r_i}{\partial t} \quad \text{--- (2)}$$

$$d\dot{r_i} = \sum_j \frac{\partial r_i}{\partial q_j}\dot{q}_j + \frac{\partial r_i}{\partial t}$$

Further displacements can be connected with δq_j

$$\Rightarrow \delta r_i = \sum_i^N \frac{\partial r_i}{\partial q_j}\delta q_j + \frac{\partial r_i}{\partial t}\cancel{\delta t}$$

Now from D-Alembert Principle

$$\Rightarrow \sum_i (\vec{F_i} - \dot{\vec{P_i}}) \cdot \delta \vec{r_i} = 0$$

$$\Rightarrow \sum_i \vec{F_i} \cdot d\vec{r_i} - \sum_i \dot{\vec{P_i}} \cdot \delta r_i = 0$$

$$\Rightarrow \sum_i F_i \cdot \frac{\partial r_i}{\partial q_j}\delta q_j - \sum_i m\ddot{r_i} \cdot \frac{\delta r_i}{\partial q_j}\delta q_j = 0$$

$$Q_j = \vec{F_i} \cdot \frac{\partial r_i}{\delta q_j}$$

Now Let us Assume the Term

$Q_j = \vec{F_i} \cdot \frac{\delta r_i}{\delta q_j}$ is known as Generalized Coordinate whose dimension is equal to work done then above Eq convert in the form of

$$\sum Q_j \,\delta q_j - \sum m \ddot{r}_i \cdot \frac{\delta r_i}{\delta q_j} \delta q_j = 0$$

$$\boxed{\sum Q_j - \sum_i m \ddot{r}_i \cdot \frac{\delta r_i}{\delta q_j} = 0} \quad (3)$$

$$\frac{d}{dt}\left[m\dot{r}_i \frac{\delta r_i}{\delta q_j}\right] - m\dot{r}_i \frac{d}{dt}\left(\frac{\delta r_i}{\delta q_j}\right) + \frac{\delta r_i}{\delta q_j} m\ddot{r}_i$$

$$\frac{d}{dt}\left(m\dot{r}_i \frac{\delta r_i}{\delta q_j}\right) - m\dot{r}_i \frac{d}{dt}\left(\frac{\delta r_i}{\delta q_j}\right) = \frac{\delta r_i}{\delta q_j} m\ddot{r}_i \quad (4)$$

Let us consider two term

$$\left.\begin{aligned} \frac{d}{dt}\left(\frac{\delta r_i}{\delta q_j}\right) &= \frac{\delta v_i}{\delta q_j} \\ \text{And} \quad \frac{\delta r_i}{\delta q_j} &= \frac{\delta v_i}{\delta q_j} \end{aligned}\right\} \quad (5)$$

from Eq (3), (4) And (5)

$$Q_j = \frac{d}{dt}\left(m\dot{r}_i \frac{\delta v_i}{\delta q_j}\right) - m\dot{r}_i \frac{\delta r_i}{\delta q_j} \quad (6)$$

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We know that total K.E of the system

$$T = \frac{1}{2} m v^2$$
$$\delta T = \frac{1}{2} m \cdot 2 v \, \delta v$$
$$\delta T = m v \, dv$$
$$\delta T = m \dot{q} \, dv$$

$$\boxed{Q_j = \frac{d}{dt}\left(\frac{\delta T}{\delta \dot{q}_j}\right) - \frac{\delta T}{\delta q_j}}$$

This the Equation of motion for Generalize Coordinates

Now Cases may be Arises.

Case-I Conservative System:-

Potential Energy does not depend upon Time. According to the definition of Conservative force we know that Conservative force is that force in which the Negative Gradient of Potential Energy is equal to Applied force.

$$F_i = -\text{grad } V$$
$$= -\frac{dV}{\delta r_i}$$

Now from the definition of Generalige force we know that

$$Q_j = \sum_i \bar{F} \cdot \frac{\delta r_i}{\delta q_j}$$

$$Q_j = -\sum_i \frac{dV}{\delta q_j}$$

And we know that

$$Q_j = \frac{d}{dt}\left(\frac{\delta T}{\delta \dot{q}_j}\right) - \frac{\delta T}{\delta q_j}$$

$$-\frac{\delta V}{\delta q_j} = \frac{d}{dt}\left(\frac{\delta T}{\delta \dot{q}_j}\right) - \frac{\delta T}{\delta q_j}$$

$$-\frac{\delta V}{\delta q_j} + \frac{\delta T}{\delta q_j} = \frac{d}{dt}\left(\frac{\delta T}{\delta \dot{q}_j}\right)$$

$$\frac{\delta}{\delta q_j}(T-V) - \frac{d}{dt}\left(\frac{\delta T}{\delta \dot{q}}\right) = 0$$

for conservative system $V = 0$

Now from the definition of Lagrange we know that difference of P.E and K.E is

$L = T - V$ So above eq becomes.

$$\boxed{\frac{\delta L}{\delta q_j} - \frac{d}{dt}\left(\frac{\delta L}{\delta \dot{q}}\right) = 0}$$

This is Lagrange eq of motion to the conservative system.

Case-II for non conservative system:-

In a non conservative system the potential energy is the function of time the value of generalized force can be written as.

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$$Q_j = -\frac{\delta V}{\delta q_j} + \frac{d}{dt}\left(\frac{\delta V}{\delta \dot{q}_j}\right)$$

$$\frac{d}{dt}\left(\frac{\delta T}{\delta \dot{q}_j}\right) - \frac{\delta T}{\delta q_j} = -\frac{\delta V}{\delta q_j} + \frac{d}{dt}\left(\frac{\delta V}{\delta \dot{q}_j}\right)$$

$$\frac{d}{dt}\frac{\delta}{\delta \dot{q}_j}(T-V) - \frac{\delta}{\delta q_j}(T-V) = 0$$

$$\boxed{\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{q}_j}\right) - \frac{\delta L}{\delta q_j} = 0}$$

from above 9t is clear that the Lagrangian Eq of Motion is Applied in both System. Either its is conservative or Non conservative